

The Algorithmic Information Content for randomly perturbed systems*

Claudio Bonanno[†]

Abstract

In this paper we prove estimates on the behaviour of the Kolmogorov-Sinai entropy relative to a partition for randomly perturbed dynamical systems. Our estimates use the entropy for the unperturbed system and are obtained using the notion of Algorithmic Information Content. The main result is an extension of known results to study time series obtained by the observation of real systems.

1 Introduction

In this paper we are interested in studying the behaviour of the Kolmogorov-Sinai (KS) entropy for randomly perturbed dynamical systems. As it has been proved in [17] for a typical situation, the KS entropy of a dynamical system perturbed by a sequence of random variables is infinite for all $\sigma > 0$; here σ is a real parameter measuring the size of the perturbation. This result shows that the KS entropy is not the right quantity to look at in randomly perturbed dynamical systems. In [17] it has been suggested to study the KS entropy relative to a partition.

We study the perturbation of a dynamical system (X, μ, f) by some noise. By noise we mean a discrete stochastic processes of independent and identically distributed (i.i.d.) random variables $\{w_n\}_{n \in \mathbb{N}}$ defined on a probability space (W, \mathcal{P}, P) with values on the interval $[-\sigma, \sigma]$, for a positive real number σ . The number σ is the parameter that measures the size of the perturbation on the original system.

We also assume that for all σ there is an ergodic invariant probability measure μ_σ on the perturbed system (X, f_σ) such that the probability measure μ is the weak limit of the sequence μ_σ as $\sigma \rightarrow 0$ (for the formal definition of μ_σ see Section 4).

In [17] it has been proved that, under these hypothesis, it holds

$$\limsup_{\sigma \rightarrow 0^+} h_{\mu_\sigma}(f_\sigma, Z) \leq h_\mu(f, Z) \leq h_\mu(f) \quad (1)$$

*AMS subject classification: 28D20, 37A50, 37M25, 60H40

[†]Dipartimento di Matematica, Università di Pisa, via Buonarroti 2/a, 56127 Pisa (Italy), email: bonanno@dm.unipi.it

where $h_\mu(f, Z)$ and $h_{\mu_\sigma}(f_\sigma, Z)$ are the KS entropies relative to the partition Z of the original and the perturbed system respectively, and by $h_\mu(f)$ we denote the KS entropy of the unperturbed dynamical system. The equality holds for certain hyperbolic systems ([17]).

In equation (1) it is used the KS entropy relative to a fixed partition, avoiding the problem of having infinite KS entropy, and it is performed the limit for the size of the perturbation going to zero. This result shows that for an unperturbed dynamical system, the KS entropy relative to a partition is well approximated by the same quantity for the perturbed system, when the perturbation is small enough.

Unfortunately when we want to study time series obtained by measurements of real systems, it is impossible to know how small is the perturbation to the original system. Hence it is impossible to perform the limit in equation (1).

Real systems are observed using a symbolic representation of the measurements. That is the phase space is divided into a finite number of sets, and we study the time series given by the symbols of the sets of the partition visited by the system as time grows.

The main idea to study real systems is to look at the system using different scales of observation, which corresponds to use different sizes for the partition of the phase space. When the partition is coarse the effects of the noise are negligible, whereas, when the partition is very fine, then the effects of the noise hide the dynamics of the original system. Then the right approach to randomly perturbed systems is to study the behaviour of the KS entropy while the partition is refined and the noise is fixed.

In this paper we prove that, when the diameter of the partition is big with respect to the size of the noise, the KS entropy of the perturbed system is a good approximation for the unperturbed KS entropy. Our main result is contained in equations (14) and (15), which in particular imply equation (1). Moreover we obtain precise estimates for the effects of the noise, in order to distinguish between the perturbation and the dynamics of the system, a very important problem in the analysis of time series ([9]). We remark that the main extension of the result contained in equation (1) is its applicability to time series obtained by observations of real systems.

To apply our method we have to face the problem of the measurement of the KS entropy in real systems (that is the KS entropy of observed time series). This is an important problem (for a review see for example [1]), and many papers have been dedicated to the approximation of the KS entropy by different indicators. We recall the approach of Grassberger and Procaccia [15] based on the generalized Renyi entropy, and that of Cohen and Procaccia [11] based on the correlation integral for a time series ([16]). In this paper we use a method introduced in [2] and [4], which is related to the notion of *Algorithmic Information Content (AIC)* (see Section 3 for the definitions).

Moreover, when applying theoretical methods to real time series, it is

always questionable whether all the hypothesis needed for the methods are verified by the single observations we study. This is this case, for example, for ergodicity that we suppose to be verified.

In the following section, we recall some basic definitions and results related to the KS entropy and its computation for dynamical systems and purely random systems. In Sections 4 and 5 we apply the method of the AIC to randomly perturbed systems, and finally in Section 6 we show some numerical experiments performed on the logistic map at the chaotic parameter $\lambda = 4$.

2 Basic definitions and results

We now briefly recall the basic definitions we need. Let (X, \mathcal{B}, μ) be a probability space, where X is a compact metric space and \mathcal{B} is the Borel σ -algebra. In this paper we restrict our attention on one dimensional spaces X , but we believe that the techniques used can be generalized to higher dimensions. Let $f : X \rightarrow X$ be a continuous map, invariant with respect to μ and ergodic. Given a finite measurable partition $Z = \{I_i\}_{i=1, \dots, N}$ of the space X , the *entropy* $H_\mu(Z)$ of the partition is defined as

$$H_\mu(Z) = - \sum_{i=1}^N \mu(I_i) \log(\mu(I_i)) \quad (2)$$

Let $f^{-1}Z$ be the partition given by the counter images $f^{-1}I_i$. Then let

$$Z_n = Z \vee f^{-1}Z \vee \dots \vee f^{-n+1}Z \quad (3)$$

be the partition given by the sets of the form

$$I_{i_0} \cap f^{-1}I_{i_1} \cap \dots \cap f^{-n+1}I_{i_{n-1}}$$

varying I_{i_j} among all the sets of Z . Then the *Kolmogorov-Sinai entropy* $h_\mu(f, Z)$ relative to the partition Z is defined as the limit

$$h_\mu(f, Z) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(Z_n) \quad (4)$$

The *Kolmogorov-Sinai entropy* $h_\mu(f)$ of the dynamical system (X, μ, f) is defined as

$$h_\mu(f) = \sup\{h_\mu(f, Z) / Z \text{ finite partition}\} \quad (5)$$

Moreover for the explicit computation of the KS entropy, the Kolmogorov-Sinai theorem says that there are some special partitions, called *generating*, for which the supremum among all the partitions is realized. Hence it is enough to compute the KS entropy relative to a generating partition to obtain the KS entropy of the system. The existence of a generating partition for dynamical systems is given by the following theorem:

Theorem 2.1 (Krieger Generator Theorem [19]). *Let (X, μ, f) be an ergodic dynamical system on a Lebesgue space X , such that the probability measure μ is invariant and $h_\mu(X, f) < \infty$. Then there is a finite generating partition for f .*

Let ϵ be the diameter of a uniform partition Z of the space X , and simply denote by $h_\mu(\epsilon)$ the KS entropy $h_\mu(f, Z)$ relative to the partition Z . This notation has the aim to enhance the role of the diameter of the partition considered. The quantity $h_\mu(\epsilon)$ is also called the ϵ -entropy of the dynamical system (see [14] for a review). The Krieger Generator Theorem implies that, if $h_\mu(f)$ is finite, there exists ϵ_0 such that $h_\mu(\epsilon) = h_\mu(f)$ for all $\epsilon \leq \epsilon_0$.

It is also possible to establish the behaviour of the ϵ -entropy for discrete stochastic processes of i.i.d. random variables.

Theorem 2.2 (Gaspard-Wang [14]). *Let $\{w_n\}_{n \in \mathbb{N}}$ be a discrete stochastic process of i.i.d. random variables with values on the interval $[0, 1]$. Then as $\epsilon \rightarrow 0$ it holds*

$$h(\epsilon) \sim -\log \epsilon$$

where the entropy is computed with respect to the invariant probability measure of the system (the induced measure) absolutely continuous with respect to the Lebesgue measure.

When dealing with randomly perturbed systems, it has been shown using numerical experiments ([14],[21]) that the expected behaviour is

$$h_{\mu_\sigma}(\epsilon) := h_{\mu_\sigma}(f_\sigma, Z) \sim \begin{cases} h_\mu(\epsilon) & \text{for } \epsilon \gg \sigma \\ -\log \epsilon & \text{for } \epsilon \ll \sigma \end{cases} \quad (6)$$

where σ denotes the standard deviation of the noise (that is the square root of the variance of the random variables w_n). The same result is expected for the generalized Renyi entropy with $q = 2$ ([22]).

We recall that for randomly perturbed systems the KS entropy $h_{\mu_\sigma}(f_\sigma)$ is infinite, hence the Krieger Generator Theorem is not applicable to the system (X, f_σ) . But we assume that the unperturbed system (X, f) has finite KS entropy $h_\mu(f)$ with generating partition of diameter ϵ_0 . Hence $h_\mu(\epsilon) = h_\mu(f)$ for all $\epsilon \leq \epsilon_0$. This implies that if the size σ of the noise is small with respect to ϵ_0 , approximating $h_\mu(\epsilon)$ by $h_{\mu_\sigma}(\epsilon)$ we find a good approximation to the KS entropy of the unperturbed system.

In the following we prove (6) using the *Algorithmic Information Content (AIC)*, briefly described in the following section. We remark that the behaviour of equation (6) is of great importance since it would give us a method to estimate the size of the random perturbation on the deterministic dynamics of the system we are observing.

3 The algorithmic information content

Given a finite alphabet \mathcal{A} , let \mathcal{A}^n be the set of all words on the alphabet \mathcal{A} of length n . The intuitive meaning of quantity of information contained in a finite string $s \in \mathcal{A}^n$ is the length of the smallest message from which you can reconstruct s on some machine. Thus, formally, the information I is a function

$$I : \mathcal{A}^* = \bigcup_{n \in \mathbb{N}} \mathcal{A}^n \rightarrow \mathbb{N}$$

on the set of finite strings on a finite alphabet \mathcal{A} which takes values in the set of natural numbers.

One of the most important information function is the *Algorithmic Information Content (AIC)*. In order to define it, it is necessary to define the notion of partial recursive function. We limit ourselves to give an intuitive idea which is very close to the formal definition. We can consider a partial recursive function as a computer C which takes a program P (namely a binary string) as an input, performs some computations, and gives a string $s = C(P)$, written on the given alphabet \mathcal{A} , as an output. The *AIC* of a string s is defined as the shortest binary program P which gives s as its output, namely

$$AIC(s, C) = \min\{|P| : C(P) = s\} \quad (7)$$

We require that our computer is a universal computing machine. Roughly speaking, a computing machine is called *universal* if it can simulate any other machine. In particular every real computer is a universal computing machine, provided that we assume that it has virtually infinite memory. For a precise definition see for example [20] or [10]. We have the following theorem

Theorem 3.1 (Kolmogorov [18]). *If C and C' are universal computing machines then*

$$|AIC(s, C) - AIC(s, C')| \leq K(C, C')$$

where $K(C, C')$ is a constant which depends only on C and C' .

This theorem implies that the information content *AIC* of s with respect to C depends only on s up to a fixed constant, then its asymptotic behaviour does not depend on the choice of C . For this reason from now on we will write $AIC(s)$ instead of $AIC(s, C)$.

The shortest program which gives a string as its output is a sort of encoding of the string, and the information which is necessary to reconstruct the string is contained in the program. Unfortunately the coding procedure associated to the Algorithmic Information Content cannot be performed by

any algorithm. This is a very deep statement and, in some sense, it is equivalent to the Turing halting problem or to the Gödel incompleteness theorem. Then the Algorithmic Information Content is a function not computable by any algorithm.

Using the notion of Algorithmic Information Content it is possible to define a notion of *complexity* for infinite strings. Let ω be an infinite string on the alphabet \mathcal{A} , that is $\omega \in \Omega := \mathcal{A}^{\mathbb{N}}$. We denote by ω^n the first n symbols of the string ω . Then $\omega^n \in \mathcal{A}^n$. The complexity measures the mean quantity of information in each digit of the string ω . Formally

Definition 3.2 (Brudno [8]). The *complexity* $K(\omega)$ of an infinite string $\omega \in \Omega$ is given by

$$K(\omega) = \limsup_{n \rightarrow \infty} \frac{AIC(\omega^n)}{n}$$

Using the method of symbolic dynamics it is possible to consider the information and the complexity of the orbits of a dynamical system. Let (X, μ, f) be an ergodic dynamical system and let $Z = \{I_1, \dots, I_N\}$ be a finite measurable partition of the space X . To the partition Z it is associated the finite alphabet $\mathcal{A} = \{1, \dots, N\}$. Define a map $\varphi_Z : X \rightarrow \Omega$, where $\Omega = \mathcal{A}^{\mathbb{N}}$, by

$$(\varphi_Z(x))_j = k \iff f^j(x) \in I_k \quad (8)$$

The sequence $\varphi_Z(x) \in \Omega$ associated to a point $x \in X$ is called the *symbolic orbit* of x relative to the partition Z . Then we have the following definition.

Definition 3.3 (Brudno [8]). The *complexity* $K(x, Z)$ relative to the partition Z of the orbit with initial condition $x \in X$ is given by

$$K(x, Z) := K(\varphi_Z(x)) = \limsup_{n \rightarrow \infty} \frac{AIC((\varphi_Z(x))^n)}{n}$$

Remark 3.4. In [8] and in [12], using open covers and computable structures, notions of complexity of orbits of a dynamical system are defined that do not depend on the partition. But in this paper we are only interested in the complexity dependent on a partition for the same reason for which we study KS entropy relative to a partition.

This notion of complexity has been related to the notion of KS entropy by the following theorem.

Theorem 3.5 (Brudno [8]). Let (X, μ, f) be an ergodic dynamical system and Z be a finite measurable partition, then $K(x, Z) = h_\mu(f, Z)$ for μ -almost any $x \in X$.

Theorem 3.6 (White [23],[24]). In the same hypothesis as before, for μ -almost any $x \in X$ it holds

$$\liminf_{n \rightarrow \infty} \frac{AIC((\varphi_Z(x))^n)}{n} = h_\mu(f, Z)$$

If the AIC were a computable function, using the previous theorems we could compute the KS entropy relative to a partition. This approach can still be useful using *optimal compression algorithms*, that is algorithms which encode symbolic strings, giving an approximation of the information contained in a string, hence an approximation of its AIC. For this approach see [4], where also a new compression algorithm is presented and applied to some well known chaotic systems.

Remark 3.7. The notion of complexity for dynamical systems has been studied for some well known weakly chaotic systems ([7],[5]) for which it holds $h_\mu(f, Z) = K(x, Z) = 0$ for all the partitions, with the aim of giving a classification of these systems according to the asymptotic behaviour of the AIC of the orbits.

Remark 3.8. The notion of AIC has been linked to other indices for dynamical systems. For example it has been related to the sensitivity to initial conditions ([13]) and to the Poincaré recurrence times ([6]).

4 Estimates of the relative KS entropy

We now introduce formal definitions for random perturbations of a dynamical system (following [17]). We also give some estimates for the KS entropy relative to a partition for the perturbed system using the tool of symbolic dynamics.

Let $\{w_n\}$ be a stochastic process of i.i.d. random variables, with each w_n defined on the probability space (W, \mathcal{P}, P) with values on the interval $[-\sigma, \sigma]$, and q^σ as the induced distribution.

Definition 4.1. A *random perturbation* of the dynamical system (X, μ, f) is a family of Markov chains f_σ^n on the probability space (W, \mathcal{P}, P) with values on X and with transition probabilities given by

$$P^\sigma(x, B) := P\{f_\sigma^{n+1} \in B \mid f_\sigma^n = x\} = q^\sigma(B - f(x))$$

for all Borel sets $B \subset X$.

The meaning of this definition is that the point x is moved to the point $f(x)$ under the unperturbed dynamics and then it disperses randomly with distribution q^σ .

We will consider two different possible actions of the random perturbation (see [3]). We talk of *output noise* when the dynamics is driven only by the unperturbed map f , and the dispersion is caused by a non exact observation of the point $f(x)$. That is given a point $x \in X$, its orbit is given by $x_n = f(x_{n-1})$ and our data are measured observing the orbit $y_n = f_\sigma(x_n) = x_n + w_n$. Instead we talk of *dynamical noise* when the dispersion is intrinsic in the dynamics and it is not caused by the observation,

in this case the dynamics is driven by f_σ . So, for a point $x \in X$ we have $x_n = f_\sigma^n(x) = f(x_{n-1}) + w_n$.

Definition 4.2. A probability measure μ_σ on X is an *invariant measure* of the randomly perturbed dynamical system (X, f_σ) if

$$\mu_\sigma(B) = \int_X P^\sigma(x, B) d\mu_\sigma(x)$$

for all Borel sets $B \subset X$.

We now start from the following assumptions:

- (i) there is an ergodic dynamical system (X, μ, f) with finite KS entropy, and denote by $h_\mu(\epsilon)$ its KS entropy relative to a finite measurable partition Z of diameter ϵ ;
- (ii) we can analyze data produced by a random perturbation (X, f_σ) of the system (output or dynamical noise) with fixed σ ;
- (iii) there is an invariant and ergodic probability measure μ_σ on (X, f_σ) and the KS entropy relative to the partition Z for the randomly perturbed system is denoted by $h_{\mu_\sigma}(\epsilon)$.

To analyze our system by the Algorithmic Information Content we have to use the method of symbolic dynamics. Given the finite measurable partition $Z = \{I_1, \dots, I_N\}$ of diameter $\epsilon = \frac{1}{N}$, we can associate a symbolic orbit in $\Omega = \{1, \dots, N\}^{\mathbb{N}}$ to any orbit in the space X using the map φ_Z defined in equation (8)

In the case of the unperturbed system (X, μ, f) we denote by $\Omega' \subseteq \Omega$ the image of the map φ_Z . The set Ω' is closed with respect to the action of the usual shift map τ defined on Ω . It is also possible to define the probability measure ν on Ω' induced by φ_Z . So we can work on the ergodic dynamical system (Ω', ν, τ) with KS entropy $h_\nu = h_\mu(\epsilon)$.

Analogously we define the map $\varphi_{Z,\sigma}$ from the product space $X \times W$ to Ω in the following way: for each $x \in X$ the map $\varphi_{Z,\sigma}(x, \cdot)$ is the symbolic map for the Markov chain f_σ^n . So for every point $x \in X$ there is a set of possible symbolic orbits in $\Omega'_\sigma := \varphi_{Z,\sigma}(X \times W)$ that correspond to different realizations of the stochastic process $\{w_n\}$. Let ν_σ be the probability measure induced on Ω'_σ by the measure μ_σ . Then we study the ergodic dynamical system $(\Omega'_\sigma, \nu_\sigma, \tau)$ with KS entropy $h_{\nu_\sigma} = h_{\mu_\sigma}(\epsilon)$.

Since now we will denote by ψ and by ω the symbolic orbits in Ω' and Ω'_σ respectively.

From the previous definitions, the following propositions easily follow

Proposition 4.3. $\Omega' \subseteq \Omega'_\sigma$

Proposition 4.4. $h_\mu(\epsilon) \leq -\log \epsilon$, $h_{\mu_\sigma}(\epsilon) \leq -\log \epsilon$.

Proposition 4.5. *If $\Omega' = \Omega'_\sigma$ and ν is absolutely continuous with respect to ν_σ ($\nu \ll \nu_\sigma$) then $h_\mu(\epsilon) = h_{\mu_\sigma}(\epsilon)$.*

Proof. For ν_σ -almost any $\omega \in \Omega'_\sigma$ it holds

$$\lim_{n \rightarrow \infty} \frac{AIC(\omega^n)}{n} = h_{\mu_\sigma}(\epsilon) \quad (9)$$

thanks to Theorems 3.5 and 3.6. Hence, since for ν -almost all $\psi \in \Omega'$ it holds

$$\lim_{n \rightarrow \infty} \frac{AIC(\psi^n)}{n} = h_\mu(\epsilon) \quad (10)$$

and $\nu \ll \nu_\sigma$ then $h_\mu(\epsilon) = h_{\mu_\sigma}(\epsilon)$. \square

We conclude this section with a lower bound for the ϵ -entropy of the random perturbed system.

Proposition 4.6 (Kifer [17] Theorem 2.4). *Suppose that all transition probabilities $P^\sigma(x, \cdot)$ have bounded densities $p^\sigma(x, y) \leq K$, then $h_{\mu_\sigma}(\epsilon) \geq -\log \epsilon - \log K$.*

Hence the KS entropy of a randomly perturbed system is always infinite, since it can be computed as the limit of $h_{\mu_\sigma}(\epsilon)$ for ϵ going to 0.

5 The AIC for noisy systems

5.1 The case of output noise

We now study the behaviour of the AIC relative to a partition in randomly perturbed dynamical systems. We start with the case of output noise (see previous section) which is easier to be studied and hence it is useful to introduce the arguments we will use for the more important case of dynamical noise.

Let Ω' and Ω'_σ be the symbolic spaces of the unperturbed and the perturbed systems respectively. We will study the ϵ -entropy of the perturbed system using the AIC of the symbolic orbits, ψ and ω , of the original system and of its perturbed version.

At any step of our dynamical system the perturbation induced by the noise changes the position of the point, and so it could change the set of the partition in which the point is. So the n -th symbol of the string ω could be different from ψ_n , the corresponding symbol of the string ψ . We recall that since we are in the case of output noise, at the step $n+1$ there is no memory of the noise at the previous step. Moreover since the random variables w_n are independent, the probability p of ω_n being different from ψ_n does not depend on the step n , neither on the set of the partition we consider.

Let $H(p)$ be the KS entropy of a p -Bernoulli trial, namely the stochastic process of independent Bernoulli variables ϑ_n with parameter p . It is well known that

$$H(p) = -p \log(p) - (1-p) \log(1-p) .$$

Hence applying Theorems 3.5 and 3.6 it holds

$$\lim_{n \rightarrow \infty} \frac{H_n(p)}{n} = H(p) \quad (11)$$

where $H_n(p)$ denotes the average of the AIC over all the sequences produced by the p -Bernoulli trial.

In this framework, we have

Proposition 5.1. *If the random variables w_n have values in the interval $[-\sigma, \sigma]$, then*

$$\mathbb{E}_{\mu_\sigma}[AIC(\omega^n)] \leq \mathbb{E}_\mu[AIC(\psi^n)] + np \log \left(2 \left\lceil \frac{\sigma}{\epsilon} \right\rceil \right) + H_n(p)$$

where $\lceil \cdot \rceil$ denotes the upper integer part of a real number and $\mathbb{E}[\cdot]$ denotes the mean value.

Proof. To prove the proposition we show an algorithm to describe the string ω^n and compute the information it needs. Let assume that we know the string ψ^n . We have then just to specify the symbols of ω^n that are different from those of ψ^n . We call i_n this information. We show that for all strings ω^n and ψ^n it holds

$$i_n \leq r_n = (\#\{i / \psi_i \neq \omega_i\}) \log \left(2 \left\lceil \frac{\sigma}{\epsilon} \right\rceil \right) + H_n(p) \quad (12)$$

The algorithm to describe ω^n works as follows. First it describes a binary string $s = (s_0, \dots, s_{n-1})$, where $s_i = 0$ implies that $\omega_i = \psi_i$ and $s_i = 1$ otherwise. To this aim we need on average $H_n(p)$ bits of information, since the string s is obtained as a string of a p -Bernoulli trial.

Moreover the algorithm needs also to specify how far on the left or on the right the noise has moved the point. To this aim we need $\log \left(2 \left\lceil \frac{\sigma}{\epsilon} \right\rceil \right)$ bits of information for each symbol of ω^n different from the corresponding symbol of ψ^n . The factor $\left\lceil \frac{\sigma}{\epsilon} \right\rceil$ counts exactly how many sets of the partition Z can be covered by the effect of the noise, and the factor 2 is needed to specify whether the point is moved to the right or to the left.

Hence to specify the string ω^n we do not need more than $AIC(\psi^n) + r_n$ bits of information. If we evaluate the mean of $AIC(\omega^n)$ over the measure μ_σ then the thesis follows from the definition of the probability p . \square

Theorem 5.2. *If $h_\mu(\epsilon)$ is the KS entropy relative to a partition Z of diameter ϵ of an unperturbed dynamical system (X, μ, f) , for the ϵ -entropy $h_{\mu_\sigma}(\epsilon)$*

of the system perturbed by an output noise $\{w_n\}$ with values in the interval $[-\sigma, \sigma]$ it holds

$$h_{\mu_\sigma}(\epsilon) \leq h_\mu(\epsilon) + p \log \left(2 \left\lceil \frac{\sigma}{\epsilon} \right\rceil \right) + H(p)$$

Proof. The thesis follows from Proposition 5.1, using equations (10) and (9) for $h_\mu(\epsilon)$ and $h_{\mu_\sigma}(\epsilon)$, and using equation (11). \square

5.2 The case of dynamical noise

We now study the case of dynamical noise. We assume to have an unperturbed dynamical system (X, μ, f) and a dynamical perturbation due to a stochastic process $\{w_n\}$ of i.i.d. random variables with values in the interval $[-\sigma, \sigma]$, such that the points $x \in X$ follow orbits given by $x_{n+1} = f(x_n) + w_{n+1}$.

Let consider a partition Z of the space X in a finite number N of measurable sets $\{I_i\}$, such that the diameter of each set of the partition is $\epsilon = 1/N$. Let $\Omega = \{1, \dots, N\}^{\mathbb{N}}$ be the space of the symbolic orbits associated to the dynamical system, and denote by ω a single string in Ω'_σ , given by the measurements made on the system. That is ω is the symbolic orbit of a point $x \in X$ after the effect of the noise. To estimate the Algorithmic Information Content of the string ω from above, we show how to construct it from the knowledge of the noise and of the unperturbed dynamics.

Let x_0 denote the initial point of an orbit of the system. By the classical symbolic representation of orbits, there is a symbolic string $\psi^0 = (\psi_0^0, \psi_1^0, \dots) \in \Omega$, relative to the partition Z , associated to the point x_0 for the unperturbed dynamical system, that is $f^i(x_0) \in I_{\psi_i^0}$ for all $i \in \mathbb{N}$, and there is also the symbolic string $\omega = (\omega_0, \omega_1, \dots)$ that is given by the perturbed system. We have $\omega_0 = \psi_0^0$. In the case of dynamical noise, we have to consider the effect due to the fact that the action of the noise is not forgotten at each step. So we have to extend the knowledge of the unperturbed orbit to more than just one iteration. To find a good number of iterations to be stored, we use the approximation of the Kolmogorov-Sinai entropy $h_\mu(f, Z) = h_\mu(\epsilon)$ relative to the partition Z by the decreasing sequence $H(T^{-n}Z|Z_n)$, where $Z_n = Z \vee T^{-1}Z \vee \dots \vee T^{-n+1}Z$ and

$$H(P|Q) = - \sum_{i,j} \mu(P_i \cap Q_j) \log \frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}$$

for any two finite partitions $P = (P_i)$ and $Q = (Q_j)$. Hence for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$h_\mu(\epsilon) - \delta \leq H(T^{-n_0}Z|Z_{n_0}) \leq h_\mu(\epsilon) + \delta \quad (13)$$

Let $\delta > 0$ be fixed, and find the corresponding integer n_0 according to equation (13). Assume that for the initial point x_0 of the orbit, we know the

first n_0 digits of the sequence ψ^0 . That is we assume to know the cylinder $Z_{n_0}(x_0)$, where with this notation we mean the set of the partition Z_{n_0} containing x_0 .

Let $\tilde{x}_1 = f(x_0)$ and $x_1 = \tilde{x}_1 + w_1$. As before we denote by ψ^1 the unperturbed symbolic orbit associated to the point x_1 . We assume to know the action of the noise w_1 with respect to the partition Z_{n_0} , so we compare the strength of the noise σ with the diameter¹ ϵ_{n_0} of the partition Z_{n_0} . Moreover, to know $Z_{n_0}(\tilde{x}_1)$ we only need the symbol $\psi_{n_0}^0$, and to know $Z_{n_0}(x_1)$, once we have $Z_{n_0}(\tilde{x}_1)$, we only need the action of w_1 with respect to the partition Z_{n_0} . This is enough to obtain $\omega_1 = \psi_0^1$. So to obtain the first two symbols of the string ω it is enough the following information: $I(\psi_0^0, \dots, \psi_{n_0-1}^0)$, $I(w_1|Z_{n_0})$ and $I(\psi_{n_0}^0|\psi_0^0, \dots, \psi_{n_0-1}^0)$. We have used $I(\cdot|\cdot)$ to denote the conditional information.

At this point it is possible to iterate the argument with the string ψ^1 instead of the string ψ^0 , hence to find the symbol $\omega_2 = \psi_0^2$ using the following information: $I(w_2|Z_{n_0})$, $I(\psi_0^1, \dots, \psi_{n_0-1}^1)$ (which is given by $Z_{n_0}(x_1)$ and is indeed known) and $I(\psi_{n_0}^1|\psi_0^1, \dots, \psi_{n_0-1}^1)$.

Iterating this argument we obtain the following proposition:

Proposition 5.3. *If the random variables w_n have values in the interval $[-\sigma, \sigma]$, then for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that*

$$\mathbb{E}_{\mu_\sigma}[AIC(\omega^n)] \leq (n-1) \left[h_\mu(\epsilon) + \delta + p \log \left(2 \left\lceil \frac{\sigma}{\epsilon_{n_0}} \right\rceil \right) \right] + H_{n-1}(p) + C$$

where $\lceil \cdot \rceil$ denotes the upper integer part of a real number, the probability p is defined as in Proposition 5.1 and $\omega^n = (\omega_0, \dots, \omega_{n-1})$.

Proof. To obtain the entropy on the right hand side we only have to use equation (13), and to remark that the conditional entropy $H(T^{-n_0}Z|Z_{n_0})$ is obtained as the integral over the space of all admissible strings of the information function $I(\psi_{n_0}|\psi_0, \dots, \psi_{n_0-1})$. The constant C is the information $I(\psi_0^0, \dots, \psi_{n_0-1}^0)$ needed to start the iteration.

For the part related to the noise we repeat the same argument of Proposition 5.1. Then the action of the noise can be bounded by

$$\mathbb{E}_P[r_n] = (n-1)p \log \left(2 \left\lceil \frac{\sigma}{\epsilon_{n_0}} \right\rceil \right) + H_{n-1}(p)$$

bits of information. □

Let again $h_{\mu_\sigma}(\epsilon)$ be the ϵ -entropy of the perturbed dynamical system, hence it follows

¹In this case we cannot ask the partition Z_{n_0} to be uniform, so we consider the diameter to be the smaller diameter of the intervals that make the partition Z_{n_0} .

Theorem 5.4. *If $h_\mu(\epsilon)$ is the ϵ -entropy of an unperturbed dynamical system (X, μ, f) , for the ϵ -entropy $h_{\mu_\sigma}(\epsilon)$ of the system perturbed by a dynamical noise $\{w_n\}$ it holds: for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$h_{\mu_\sigma}(\epsilon) \leq h_\mu(\epsilon) + \delta + p \log \left(2 \left\lceil \frac{\sigma}{\epsilon_{n_0}} \right\rceil \right) + H(p)$$

5.3 Conclusions

At this point we can make some conclusions about the estimates of the KS entropy of a randomly perturbed system.

We recall the conditions we imposed in Section 4 on the data we study. So, given a fixed σ , we vary the diameter ϵ of the partition considered.

Let $\epsilon > \sigma$. In this case we can assume that for some ϵ we have $\Omega' = \Omega$. This implies (Propositions 4.3 and 4.5) that $h_{\mu_\sigma}(\epsilon) = h_\mu(\epsilon)$. When $\Omega' \subset \Omega$ we have that, fixed $\delta > 0$ such that $h_\mu(\epsilon) + \delta < -\log \epsilon$, if $\epsilon_{n_0} > \sigma$

$$-\log \epsilon - \text{const} \leq h_{\mu_\sigma}(\epsilon) \leq h_\mu(\epsilon) + \delta + p \log 2 + H(p) \quad (14)$$

using Proposition 4.6 and Theorem 5.4. We recall that $H(p)$ takes its maximum value $\log 2$ for $p = 0.5$, and it converges to 0 as p goes to 0 or 1. Then if $\epsilon_{n_0} \gg \sigma$ we have that p converges to 0, hence $(p \log 2 + H(p))$ converges to 0, and equation (14) becomes $h_{\mu_\sigma}(\epsilon) \leq h_\mu(\epsilon) + \delta$. In particular this implies equation (1). If instead $\epsilon_{n_0} \leq \sigma$ then we have the same conclusion as in the following case.

If $\epsilon < \sigma$ then it holds

$$\begin{aligned} -\log \epsilon - \text{const} &\leq h_{\mu_\sigma}(\epsilon) \leq \\ &\leq \min \left(-\log \epsilon, h_\mu(\epsilon) + \delta + p \log \left(2 \left\lceil \frac{\sigma}{\epsilon_{n_0}} \right\rceil \right) + H(p) \right) \end{aligned} \quad (15)$$

using Propositions 4.4, 4.6 and Theorem 5.4. Again if $\epsilon_{n_0} \ll \sigma$ then p converges to 1, hence $H(p)$ converges to 0, and the estimate on the right hand side of equation (15) becomes $h_\mu(\epsilon) + \delta - \log \epsilon_{n_0} + \text{const}$.

From equations (14) and (15) it emerges that by computing the KS entropy relative to partitions, we obtain an indication of the smallness of the random perturbation present in the data we are studying. Indeed if in the curve $h_{\mu_\sigma}(\epsilon)$ we find an almost flat region ($\epsilon \geq \epsilon_1$), and as ϵ decreases there is a change in the behaviour of the curve, that almost behaves as $-\log(\epsilon)$ ($\epsilon \leq \epsilon_2$), then the interval (ϵ_2, ϵ_1) should be a good approximation for the size σ of the perturbation. In the next section we apply this method to a perturbed dynamical system, using a compression algorithm to estimate the Algorithmic Information Content.

Moreover this method can also be thought of as a method to distinguish between purely stochastic systems and randomly perturbed dynamical systems. Indeed for randomly perturbed systems equations (14) and (15) suggest the presence of a change of behaviour for the ϵ -entropy at the level

of the perturbation. This change should not appear in purely stochastic systems. However this aspect has to be examined further.

6 Numerical experiments

In the previous section we have proved that the Algorithmic Information Content can be used for randomly perturbed dynamical systems to obtain information on the size of the random perturbation. We remark that unfortunately the AIC, as defined in equation (7), is not a computable function. So to compute the information content of a string one needs to use a *compression algorithm*, that is an algorithm which approximates the value of the information function AIC.

Formally, a *compression algorithm* is defined as a recursive reversible coding procedure $Z : \mathcal{A}^* \rightarrow \{0,1\}^*$ (for example, the data compression algorithms that are in any personal computer). The information content I_Z of a finite string s computed by the algorithm Z is given by the binary length of the compressed string, that is $I_Z(s) = |Z(s)|$.

Not all compression algorithms approximate the AIC. In [4] this point is discussed in details. For the purposes of this paper, it is sufficient that

$$\limsup_{n \rightarrow \infty} \frac{I_Z(\omega^n)}{n} = K(\omega)$$

where $K(\omega)$ is the complexity of an infinite string ω (see Definition 3.2). The algorithms with this property are called *optimal*.

In [4] it is also introduced a compression algorithm called *CASToRe*, which has been created with the specific aim of studying weakly chaotic dynamical systems, namely chaotic systems with null KS entropy.

Using CASToRe we have analyzed an example of randomly perturbed dynamical system. We have chosen the logistic map

$$f : x \mapsto \lambda x(1 - x)$$

on the interval $[0, 1]$ with $\lambda = 4$. To this map we have added a dynamical noise given by independent random variables $\{w_n\}$ uniformly distributed on the interval $[-\sigma, \sigma]$.

Keeping σ fixed we have varied the diameter of the partition from 0.5 to 0.004=1/250, and we have computed the complexity of the 10^6 -long symbolic orbits of the perturbed system relative to the different partitions.

In figure 1 we show the results. It is a log-linear plot of the complexity of the symbolic orbits versus the diameter of the partition, plotted on the x axis. The solid line is the upper bound $-\log \epsilon$, and the other curves are the empirical curves which correspond to the following values of σ : 0.5, 0.1, 0.02, 0.01, 0.001 from the upper to the lower, respectively. For each of the curves $h_{\mu_\sigma}(\epsilon)$ it is possible to identify the intervals (ϵ_2, ϵ_1) , that give good

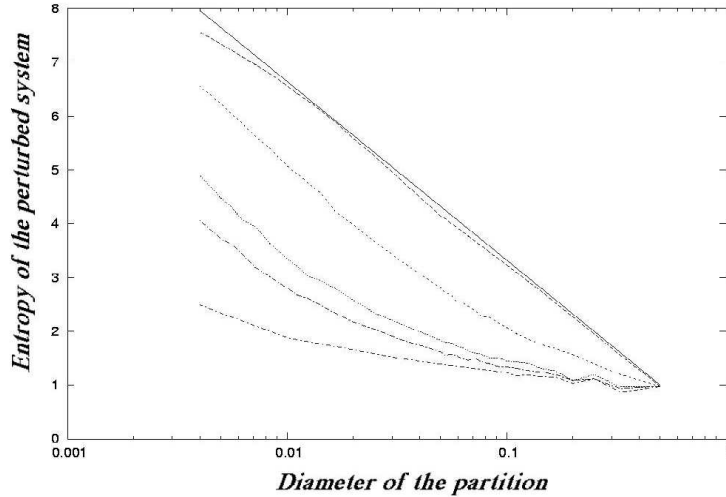


Figure 1: *The complexity for different values of σ with respect to the diameter of the partition.*

approximation of σ , in agreement with the theoretical result in equations (14) and (15) and the subsequent comments.

When the diameter of the partition is high with respect to the size of the noise, then we have a good approximation of the unperturbed KS entropy of the system, since the partition with diameter 0.5 is generating. In figure 1 it is possible to see that the complexity of the symbolic orbits for partitions with diameter close to 0.5 is close to 1 (the KS entropy² of the logistic map) for the empirical curves with $\sigma < 0.1$.

References

- [1] H.D.I.Abarbanel, R.Brown, J.J.Sidorowich, L.S.Tsimring, *The analysis of observed chaotic data in physical systems*, Rev. Mod. Phys. **65** (1993), 1331–1392
- [2] F.Argenti, V.Benci, P.Cerrai, A.Cordelli, S.Galatolo, G.Menconi, *Information and dynamical systems: a concrete measurement on sporadic dynamics*, Chaos Solitons Fractals **13** (2002), 461–469
- [3] J.Argyris, I.Andreadis, *On the influence of noise on the largest Lyapunov exponent of attractors of stochastic dynamic systems*, Chaos Solitons Fractals **9** (1998), 959–963

²The KS entropy is $\log 2$, but we use logarithms in base 2 since we measure binary information contents.

- [4] V.Benci, C.Bonanno, S.Galatolo, G.Menconi, M.Virgilio, *Dynamical systems and computable information*, Disc. Cont. Dyn. Syst. - B, to appear
- [5] C.Bonanno, *The Manneville map: topological, metric and algorithmic entropy*, <http://arXiv.org/abs/math.DS/0107195> (2001)
- [6] C.Bonanno, S.Galatolo, S.Isola, *Poincaré recurrence times and algorithmic complexity*, submitted (2003)
- [7] C.Bonanno, G.Menconi, *Computational information for the logistic map at the chaos threshold*, Disc. Cont. Dyn. Syst.- B **2** (2002), 415–431
- [8] A.A.Brudno, *Entropy and the complexity of the trajectories of a dynamical system*, Trans. Moscow Math. Soc. **2** (1983), 127–151
- [9] M.Cencini, M.Falcioni, E.Olbrich, H.Kantz, A.Vulpiani, *Chaos or noise: difficulties of a distinction*, Phys. Rev. E **62** (2000), 427–437
- [10] G.J.Chaitin, “Information, randomness and incompleteness, papers on algorithmic information theory”, World Scientific, Singapore, 1987
- [11] A.Cohen, I.Procaccia, *Computing the Kolmogorov entropy from time signals of dissipative and conservative dynamical systems*, Phys. Rev. A **31** (1985), 1872–1882
- [12] S.Galatolo, *Orbit complexity by computable structures*, Nonlinearity **13** (2000), 1531–1546
- [13] S. Galatolo, *Complexity, initial data sensitivity, dimension and weak chaos in dynamical systems*, Nonlinearity **16** (2003), 1219–1238
- [14] P.Gaspard, X.J.Wang, *Noise, chaos, and (ϵ, τ) -entropy per unit time*, Phys. Rep. **235** (1993), 291–343
- [15] P.Grassberger, I.Procaccia, *Estimation of the Kolmogorov entropy from a chaotic signal*, Phys. Rev. A **28** (1983), 2591–2593
- [16] P. Grassberger, I. Procaccia, *Characterization of strange attractors*, Phys. Rev. Lett. **50** (1983), 346–349
- [17] Y.Kifer, “Random perturbations of dynamical systems”, Birkhäuser, 1988
- [18] A.N.Kolmogorov, *Combinatorial foundations of information theory and the calculus of probabilities*, Russ. Math. Surv. **38** (1983), 29–40
- [19] W.Krieger, *On entropy and generators of measure-preserving transformations*, Trans. Amer. Math. Soc. **149** (1970), 453–464

- [20] M.Li, P.Vitanyi, “An introduction to Kolmogorov complexity and its applications”, Springer, 1993
- [21] C.Schittenkopf, G.Deco, *Identification of deterministic chaos by an information theoretic measure of the sensitive dependence on the initial conditions*, Physica D **110** (1997), 173–181
- [22] T.Schreiber, H.Kantz, *Noise in chaotic data: diagnosis and treatment*, Chaos **5** (1995), 133–142
- [23] H.White, “On the algorithmic complexity of trajectories of points in dynamical systems”, Ph.D. dissertation Univ. of North Carolina at Chapel Hill, 1991
- [24] H.White, *Algorithmic complexity of points in a dynamical system*, Ergod. Th. Dynam. Syst. **13** (1993), 807–830